ANALYSIS OF TURBULENT HEAT TRANSFER DURING NATURAL CONVECTION

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The cube-root law and the limiting square-root law of heat transfer during natural convection with developed turbulence are derived analytically.

Heat transfer during natural convection is, as we well know, characterized by self-adjointness with respect to the governing dimension and is described by the cube-root law. This law has been established experimentally in [1], with the value of GrPr $\sim 10^{13}$ attained on the basis of the governing dimension (diameter of a sphere) equal to 16 m, and it would be interesting to also derive it analytically.

We will consider the heat transfer during natural convection at a flat vertical plate, with the convective stream sufficiently turbulent. The equation of the steady-state (average) shear stress profile across the boundary layer is

$$\frac{\partial \tau_y}{\partial y} = -\rho\left(\overline{W}_x \frac{\partial \overline{W}_x}{\partial x} + \overline{W}_y \frac{\partial \overline{W}_y}{\partial y}\right) + g\rho\beta(T - T_a), \tag{1}$$

where the terms on the right-hand side represent the inertia forces and the convection (lift) force respectively.

The variation of the turbulent thermal flux across the boundary layer is expressed by the equation

$$\frac{\partial q_y}{\partial y} + c_p\rho\left(\overline{W}_x \frac{\partial T}{\partial x} + \overline{W}_y \frac{\partial T}{\partial y}\right) = 0. \tag{2}$$

We now define τ_y and q_y in terms of the following functions

$$\tau_y = \tau_0(1-u)f(u) = \tau_0(1-u)(a_0 + a_1 u + a_2 u^2), \tag{3}$$

$$q_y = q_0(1-u)F(u) = q_0(1-u)(b_0 + b_1 u + b_2 u^2), \tag{4}$$

with τ_0 and q_0 denoting respectively the shear friction and the thermal flux at the surface.

Analogous functions for forced convection were introduced by G. S. Moroz and by Pohlhausen [2].

We next assume that the thermal flux decreases along the normal coordinate exponentially. For this, we represent the dimensionless argument of the thermal flux function in exponential form, letting

$$u = 1 - \exp\left(-\varkappa \frac{y}{\delta}\right), \tag{5}$$

where $\varkappa$ is a constant and δ denotes that part of the boundary-layer thickness which corresponds to a positive velocity increment (gradient) normal to the surface. Then u will vary from 0 at the surface (y = 0) to 1 (y = ∞), i.e., the selected function is convenient in that its integration limits are from 0 to ∞. According to the Reynolds analogy, the same assumption will be made for the shear stress profile.

The number of terms in the expansion depends on the boundary conditions:

1. When $\tau = \tau_0$ at y = 0 and u = 0, then (3) and (1) yield $a_0 = 1$
2. When $\tau = \tau_0$ at y = 0 and u = 0, then (1) and (5) yield (6)

$$a_1 = 1 + \frac{g\rho\beta\delta(T_s - T_a)}{\varkappa\tau_0} = 1 + A_s; \tag{7}$$

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3) When $\tau_y = 0$ at $y = \delta$ and $u = u_0 = 1 - \exp(-\varkappa)$ then (3) yields for the second coefficient (8)

$$a_2 = -\frac{1 - a_1 u_0}{u_0^2}. \tag{9}$$

In order to determine the polynomial coefficients in (4), we proceed as before:

1. When $q = q_0$ at $y = 0$ and $u = 0$, then (4) $b_0 = 1$; (10)

2. When $q = q_0$ at $y = 0$ and $u = 0$, then (2), (5), and (4) yield $b_1 = 1$; (11)

3) When $(\partial^2 q_y/\partial u^2)_{y=0} = 0$ at $y = 0$ and $u = 0$, then $b_2 = 1$. (12)

The first integral expression in the variable u, for the thermal flux normal to the wall, is

$$q_0 = \frac{\partial}{\partial x} \int_0^1 c_p \rho \overline{W}_x (T_y - T_a) \frac{du}{1-u} \frac{\delta}{\varkappa} \tag{13}$$

or in terms of the Nusselt number, after both sides of Eq. (13) have been multiplied by l/λ and with $x/l = \xi$, $\delta/\varkappa = \delta^*$, $a = \lambda/c_p\rho$,

$$\Psi = \frac{d}{d\xi}\left[\frac{\delta^*}{a} \int_0^1 \overline{W}_x (T_y - T_a) \frac{du}{1-u}\right]. \tag{14}$$

Expression (14) is very important: it is essentially the Nusselt relation.

The second integral expression can be obtained from the relation between stress τ and the temperature; in terms of variable u we have

$$\tau_0 = g\rho\beta \int_{0l}^1 (T_y - T_a) \frac{\delta}{\varkappa} \frac{du}{1-u} - \int_0^1 \rho \frac{\partial \overline{W}_x^2}{\partial x} \frac{\delta}{\varkappa} \frac{du}{1-u}. \tag{15}$$

In order to determine the temperature profile across the boundary layer, we use the relation

$$q_y = -\overline{\lambda}_T \frac{\partial T}{\partial y} \tag{16}$$

and find the relation between $(T_y - T_a)$ and $T_s - T_a)$:

$$(T_y - T_a) = (T_s - T_a) \frac{\alpha\delta}{\varkappa} \int_u^1 \frac{F(u)}{\overline{\lambda}_T} du. \tag{17}$$

Inserting for $(T_y - T_a)$ expression (17) into (15), with

$$T_s - T_a = \frac{q_0 \delta}{\varkappa} \int_0^1 \frac{F(u)}{\overline{\lambda}_T} du,$$

we obtain

$$\Psi = \frac{q_0 l}{\lambda} = \frac{l\varkappa (T_s - T_a)}{\lambda\delta \int_0^1 \frac{F(u)}{\overline{\lambda}_T} du}. \tag{18}$$

It follows from the Prandtl–Karman theory that the velocity profile across the boundary layer is most closely described by a "logarithmic" law, but this would be mathematically very difficult to apply in the analysis here and, therefore, we used instead the Karman power-law relation

$$\overline{W}_x = B_p \sqrt{\frac{\tau_0}{\rho}} \left(\sqrt{\frac{\tau_0}{\rho}} \frac{y}{\nu}\right)^p, \tag{19}$$

where B_p and p are constant numbers.

For the turbulent viscosity as a function of y we have

$$\overline{\mu}_T = \rho \frac{\sqrt{\frac{\tau_0}{\rho}}}{pB_p} \left(\sqrt{\frac{\tau_0}{\rho}} \frac{\delta}{\nu}\right)^{-p} \delta\eta^{1-p}, \tag{20}$$

with $\eta = y/\delta$.

Assuming, in accordance with (5) that u ≈ $\varkappa(y/\delta)$ for small values of u (inasmuch as only a small part of the boundary layer directly adjacent to the wall surface plays an important role in the heat transfer), we have $\eta = u/\varkappa$ and, letting $\delta^*/\varkappa = \delta^*$, we obtain

$$\bar{\mu}_T = \rho \frac{\sqrt{\frac{\tau_0}{\rho}}}{pB_p} \left(\sqrt{\frac{\tau_0}{\rho}} \frac{\delta^*}{\nu}\right)^{-p} \delta^* u^{1-p} = \bar{\mu}_0 u^{1-p}. \tag{21}$$

For the kinematic viscosity we assume a similar relation:

$$\bar{\nu}_T = \bar{\nu}_0 u^{1-p}. \tag{22}$$

Here, according to (21),

$$\bar{\nu}_0 = \frac{\bar{\mu}_0}{\rho} = \frac{\nu}{pB_p}\left(\sqrt{\frac{\tau_0}{\rho}} \frac{\delta^*}{\nu}\right)^{1-p}. \tag{23}$$

In the turbulent mode the thermal conductivity is related to the dynamic viscosity as follows:

$$\bar{\lambda}_T = c_p \bar{\mu}_T. \tag{24}$$

With (21) we obtain now

$$\bar{\lambda}_T = \frac{\rho c_p \nu}{pB_p}\left(\sqrt{\frac{\tau_0}{\rho}} \frac{\delta^*}{\nu}\right)^{1-p} u^{1-p} \tag{25}$$

or, with (23),

$$\frac{\bar{\lambda}_0}{\lambda} = \Pr \frac{\bar{\nu}_0}{\nu} = \Pr \omega. \tag{26}$$

Now, in terms of (26), formula (17) becomes

$$T_y - T_a = (T_s - T_a) \frac{\mathrm{Nu}^*}{\omega \Pr} \int_0^1 F(u) u^{p-1} du, \tag{27}$$

with

$$\mathrm{Nu}^* = \frac{\alpha \delta^*}{\lambda}. \tag{28}$$

A transformation of the integral in (2), with $T_y = T_s$ at u = 0, yields

$$\mathrm{Nu}^* = p\omega \Pr \frac{1}{J_0(p)}. \tag{29}$$

Then

$$T_y - T_a = \frac{T_s - T_a}{J_0(p)}\left[(1-u^p) + \frac{p}{p+1}(1-u^{p+1}) + \frac{p}{p+2}(1-u^{p+2})\right] \tag{30}$$

(the temperature profile in Fig. 1 has been plotted according to Eq. (30) in $(T_y - T_a)/(T_s - T_a)$, y/δ^* coordinates).

Now the velocity component $\bar{W}_x$ will be expressed in terms of variable u and the other parameters.

According to the definition of shear friction

$$\tau_y = \bar{\mu}_T \frac{\partial \bar{W}_x}{\partial y}^* , \tag{31}$$

but from (23) we have

$$\bar{\mu}_T = \rho \nu \omega u^{1-p} \tag{32}$$

or with the aid of (5)

$$\bar{W}_x = \frac{\tau_0 \delta^*}{\rho \nu \omega} \int u^{p-1} f(u) du + C \tag{33}$$

*P. L. Kapitsa [4] has suggested the possibility that the quantity $W_y' l'$ increases continuously from the rigid wall surface on.

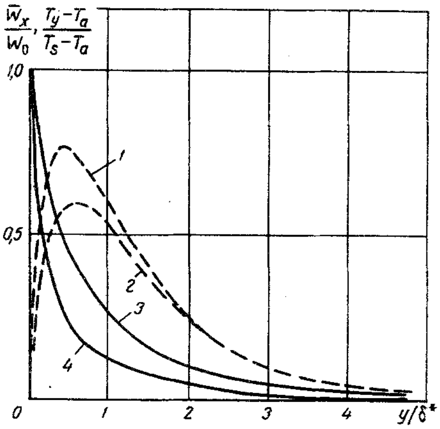

Fig. 1. Temperature and velocity profiles plotted according to formulas (30) and (34) respectively: $(T_y - T_a)/(T_S - T_a) = f(y/\delta^*)$ (solid lines), W_x/W_0 (dashed line) with $p = 1/7$ (1, 4), $p = 1/3$ (2, 3).

For $\overline{W}_x = 0$ at $u = 0$ there follows $C = 0$.

As before, we transform the integral in (33)

$$\overline{W}_x = W_0 \frac{u^p}{p}(1-u)\left(1 - \frac{p}{p+2}a_2 u\right), \tag{34}$$

with $W_0 = \delta^* \tau_0/\rho\nu\omega$ (the velocity profile in Fig. 1 has been plotted according to Eq. (34) in $\overline{W}_x/W_0$, y/δ^* coordinates).

Inserting now expression (34) for $\overline{W}_x$ into (14) and using expression (3), we obtain

$$\Psi = \frac{d}{d\xi}\left[\frac{W_0\delta^*(T_s - T_a)}{paJ_0(p)} \int_0^1 u^p \left(1 - \frac{p}{p+2}a_2 u\right)\Phi(u)\, du\right]. \tag{35}$$

When $T_S - T_a$ = const., then $\Psi/T_S - T_a$ = Nu or from (35) and (34)

$$\mathrm{Nu} = \frac{d}{d\xi}\left[\frac{\left(\sqrt{\frac{\tau_0}{\rho}}\right)^2}{\nu^2} \Pr \frac{1}{\omega p} \frac{J_3(p)}{J_0(p)}\right], \tag{36}$$

However,

$$\frac{\sqrt{\frac{\tau_0}{\rho}}\,\delta^*}{\nu} = \mathrm{Re}^* \tag{37}$$

and

$$\omega p = \frac{1}{B_p}(\mathrm{Re}^*)^{1-p}. \tag{38}$$

Then formula (36) becomes

$$\mathrm{Nu} = \frac{d}{d\xi}\left[(\mathrm{Re}^*)^{1+p} \Pr B_p \frac{J_3(p)}{J_0(p)}\right]. \tag{39}$$

We will further seek a relation between Re* and Gr*. Rearranging Eq. (15) for τ_0, with relation (30) for $T_y - T_a$ and relation (34) for $\overline{W}_x$, we obtain

$$\tau_0 = g\rho\beta\delta^* \frac{T_s - T_a}{J_0(p)} J_1(p) - \rho \frac{\partial}{\partial x}\left[\delta^* \left(\frac{\tau_0\delta^*}{\rho\nu\omega p}\right)^2 J_2(p)\right]. \tag{40}$$

Inserting Re* from (37) and (38) into (40), with $\mathrm{Gr}^* = g\beta\Delta T_S \delta^{*3}/\nu^2$ and $\delta^*/1 = \delta_S$, yields

$$\mathrm{Re}^{*2} = \mathrm{Gr}^* \frac{J_1(p)}{J_0(p)} - \delta_s \frac{d}{d\xi}\left[\frac{B_p^2}{\delta_s} \mathrm{Re}^{*2(1+p)} J_2(p)\right]. \tag{41}$$

We will now confine our analysis to the approximate solution, disregarding the second term in Eq. (14), and let

$$\mathrm{Gr}^* = \mathrm{Gr}\,\delta_s^3, \tag{42}$$

which then yields

$$\mathrm{Re}^{*2} = \mathrm{Gr}\,\frac{J_1(p)}{J_0(p)}\,\delta_s^3. \tag{43}$$

We now try to relate the Nusselt number and the Grashof number. According to (29) and (38),

$$\mathrm{Nu}^* = \frac{1}{B_p J_0(p)}\,\mathrm{Re}^{*(1-p)}\mathrm{Pr}, \tag{44}$$

but, since

$$\mathrm{Nu}^* = \frac{\alpha\delta^*}{\lambda} = \frac{\alpha l \delta_s}{\lambda} = \mathrm{Nu}\,\delta_s \tag{45}$$

and, according to (43),

$$\delta_s = \mathrm{Re}^{*\frac{2}{3}}\,\mathrm{Gr}^{-\frac{1}{3}}\left[\frac{J_0(p)}{J_1(p)}\right]^{\frac{1}{3}}, \tag{46}$$

hence

$$\mathrm{Nu} = \frac{1}{B_p J_0(p)\,\delta_s}\,\mathrm{Re}^{*(1-p)}\,\mathrm{Pr} = \frac{\mathrm{Pr}}{B_p J_0(p)}\left[\frac{J_1(p)}{J_0(p)}\right]^{\frac{1}{3}}\mathrm{Gr}^{\frac{1}{3}}\,\mathrm{Re}^{*\frac{1-3p}{3}}. \tag{47}$$

On the other hand, for the Nusselt number we have Eq. (39) and will use it for expressing Re* in terms of the Grashof number and the other parameters. Finally, (39) and (47) yield

$$\mathrm{Re}^{*(1-p)}\,\frac{1}{\delta^*} = \frac{d}{d\xi}\left[\mathrm{Re}^{*(1+p)}\,B_p^2 J_3(p)\right] \tag{48}$$

or, considering (46), the differential equation

$$\frac{d}{d\xi}\left[\mathrm{Re}^{*(1+p)}\right] = \frac{1}{B_p^2 J_3(p)}\left[\frac{J_1(p)}{J_0(p)}\right]^{\frac{1}{3}}\mathrm{Gr}^{\frac{1}{3}}\,\mathrm{Re}^{*\frac{1-3p}{3}}. \tag{49}$$

Integrating (49) yields (with $\mathrm{Re}^{*(1+p)} = z$ and inasmuch as $\mathrm{Re}^* \to 0$ and $C = 0$ when $\xi \to 0$)

$$\mathrm{Re}^{*\frac{2+6p}{3}} = \frac{2+6p}{3(1+p)}\,\frac{1}{B_p^2 J_3(p)}\left[\frac{J_1(p)}{J_0(p)}\right]^{\frac{1}{3}}\mathrm{Gr}^{\frac{1}{3}}\xi = A_p\,\mathrm{Gr}^{\frac{1}{3}}\,\xi \tag{50}$$

or

$$\mathrm{Re}^* = A_p^{\frac{3}{2+6p}}\,\mathrm{Gr}^{\frac{1}{2+6p}}\,\xi^{\frac{3}{2+6p}}, \tag{51}$$

where

$$A_p = \frac{2+6p}{3(1+p)\,B_p^2 J_3(p)}\left[\frac{J_1(p)}{J_0(p)}\right]^{\frac{1}{3}}. \tag{52}$$

Inserting Re* from (51) into (47) yields

$$\mathrm{Nu} = \bar{A}_p\,\mathrm{Pr}\,\mathrm{Gr}^{\frac{1+p}{2(1+3p)}}\,\xi^{\frac{1-3p}{2+6p}}, \tag{53}$$

where

$$\bar{A}_p = \frac{1}{B_p J_0(p)}\left[\frac{J_1(p)}{J_0(p)}\right]^{\frac{1}{3}} A_p^{\frac{1-3p}{2+6p}}. \tag{54}$$

An analysis of formula (53) leads to the following conclusions:

1. If the heat transfer coefficient is to be independent of the linear dimension, then the power exponent of ξ must be equated to zero, i.e., $1-3p/2+6p = 0$ and thus $p = 1/3$, but then (53) becomes

$$\mathrm{Nu} = \bar{A}_p\,\mathrm{Pr}\,\mathrm{Gr}^{\frac{1}{3}}. \tag{55}$$

Obviously, this formula corresponds to the law of "self-adjointness" with respect to the dimension. In other words, developed thermal turbulence prevails when p = 1/3 (while forced turbulence is, according to Karman, characterized by p = 1/7).

2. In order to satisfy the limiting relation $Nu = f(Gr^{1/2})$, one must let $1 + p/2(1 + 3p) = 1/2$, i.e., p = 0 and, consequently, the power exponent of ξ becomes equal to 1/2. The Nusselt number then becomes proportional to the square root of the linear dimension, which is in agreement with the gist of the Frank—Kamenetskii corollary [3], i.e.,

$$Nu = \bar{A}_p \Pr Gr^{\frac{1}{2}} \xi^{\frac{1}{2}}. \tag{56}$$

3. It is interesting that, with p = 1, formula (53) reduces to the laminar fourth-root law:

$$Nu = \bar{A}_p \Pr Gr^{\frac{1}{4}} \xi^{-\frac{1}{4}}, \tag{57}$$

where the negative exponent of ξ represents the well known decrease in the heat transfer coefficient with increasing distance from the origin of convective flow.

A further analysis of formula (53) makes it feasible to consider the case of Pr = 1 only, to include the Prandtl number in the coefficient $\bar{A}_p$, with the latter regarded as a function of the Prandtl number $\bar{A}_p = f(\Pr)$, and to calculate the values of this coefficient according to the formula

$$\bar{A}_p = \frac{1}{B_p J_0(p)} \left[\frac{J_1(p)}{J_0(p)}\right]^{\frac{1}{3}} \left\{\frac{2+6p}{3(1+p)} \frac{1}{B_p^2 J_3(p)} \left[\frac{J_1(p)}{J_0(p)}\right]^{\frac{1}{3}}\right\}^{\frac{1-3p}{2+6p}}. \tag{58}$$

For p = 1/3, 1/7, and 1/10 we have respectively

$$p = \frac{1}{3};\ J_1 = 1.674;\ J_0 = 1.392;\ J_3 = 0.131;\ a_2 = -10.2;$$

$$p = \frac{1}{7};\ J_1 = 1.658;\ J_0 = 1.190;\ J_3 = 0.106;\ a_2 = -18.1;$$

$$p = \frac{1}{10};\ J_1 = 1.524;\ J_0 = 1.138;\ J_3 = 0.065;\ a_2 = -24.2.$$

For eliminating a_2 we have used the formula

$$a_2 = -\left\{1 + \frac{p}{p+1}\left[1 + \frac{J_0(p)}{J_1(p)}\right]\right\} \frac{2+p}{p}. \tag{59}$$

Since for an approximate determination of τ_0 we have used Eq. (41) without the second term, hence from (7) we obtain

$$a_1 = 1 + \frac{J_0(p)}{J_1(p)}. \tag{60}$$

Inserting into (58) the found values of $J_1(p)$, $J_0(p)$, and $J_3(p)$ yields

$$\bar{A}_{p=\frac{1}{3}} = 0.765\, B_p^{-1};\ \bar{A}_{p=\frac{1}{7}} = 1.452\, B_p^{-\frac{7}{5}};\ \bar{A}_{p=\frac{1}{10}} = 1.95\, B_p^{-\frac{20}{13}}. \tag{61}$$

With $B_p = 8.74$, to the first approximation according to Karman, we obtain

$$\bar{A}_{p=\frac{1}{3}} = 0.0876 \simeq 0.1;\ \bar{A}_{p=\frac{1}{7}} = 0.697;\ \bar{A}_{p=\frac{1}{10}} = 0.696. \tag{62}$$

The first approximation $\bar{A}_{p=1/3}$, ~0.10 obtained on the basis of the coefficient B_p for forced convection is sufficiently close to the test value of $\bar{A}_{p=1/3} = 0.13$ based on natural convection. This makes it feasible now, in turn, to determine B_p on the basis of the test value for B_p according to the cube-root law: $B_p = 0.765/0.13 = 5.88$.

It is to be noted that these values indicate a close structural similarity between natural and forced turbulent flow.

NOTATION

x, y is the longitudinal and normal coordinate respectively;
τ_y is the tangential component of turbulent friction in a layer at a distance y from the plate surface;

$q_y = -\bar{\lambda}_t(\partial T/\partial y)$ is the turbulent thermal flux per unit area per unit time, normal to the surface, from a layer at a distance y from the plate surface;
T is the instantaneous temperature of the fluid;
T_a is the temperature of the fluid at infinity (far from the plate);
T_s is the temperature of the plate;
T_y is the temperature of a layer at a distance y from the plate surface;
ρ is the density of the fluid;
c_p is the specific heat of the fluid;
β is the thermal expansivity of the fluid;
$\bar{u}_t$ is the turbulent (average) dynamic viscosity;
$\bar{\lambda}_t$ is the turbulent (average) thermal conductivity;
α is the coefficient of heat transfer during natural convection;
$\overline{W}_x, \overline{W}_y$ are the longitudinal and normal component of turbulent (average) velocity;

$$f(u) = a_0 + a_1 u + a_2 u^2; \quad F(u) = b_0 + b_1 u + b_2 u^2;$$

$$\Phi(u) = \left[(1-u^p) + \frac{p}{p+1}(1-u^{p+1}) + \frac{p}{p+2}(1-u^{p+2})\right]; \quad J_0(p) = 1 + \frac{p}{p+1} + \frac{p}{p+2}; \quad J_1(p) = \int_0^1 \left[(1-u^p) + \frac{p}{p+1}(1-u^{p+1}) + \frac{p}{p+2}(1-u^{p+2})\right]\frac{du}{1-u}^{*};$$

$$J_2(p) = \int_0^1 u^{2p}(1-u)\left(1 - \frac{p}{p+2}a_2 u\right)^2 du; \quad J_3(p) = \int_0^1 u^p\left(1 - \frac{p}{p+2}a_2 u\right)\Phi(u)\,du.$$

$$J_1(p) = \int_0^1 \left[(1-u^p) + \frac{p}{p+1}(1-u^{p+1}) + \frac{p}{p+2}(1-u^{p+2})\frac{du}{1-u}\right]^{*)}.$$

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*The relation between this function and the Γ(p)-function makes it possible to use tables for calculations).